JOURNAL OF APPROXIMATION THEORY **61**, 118–130 (1990)

Construction of the Best Monotone Approximation on $L_p[0, 1]$

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Communicated by R. Bojanic

Received October 27, 1987

1. INTRODUCTION

For $1 \leqslant p < \infty$, let L_p denote the Banach space of pth power Lebesgue integrable functions on $[0, 1]$ with $\|f\|_p = (\int_0^1 |f|^p)^{1/p}$. Let M_p denote the set of nondecreasing functions in L_p. For $1 < p < \infty$, each $f \in L_p$ has a unique best approximation from M_p, while, for $p = 1$, existence of a best approximation from M_1 follows from Proposition 4 of [6].

Recently, there has been interest in characterizing best L_1 approximations from M_1 [1–4, 8]. The approach, in most instances, was measure theoretic. In [8], a duality approach was used to extend the results to all L_p, $1 \leqslant p < \infty$.

In a recent paper [4] an explicit construction was given for a best L_1 approximation to f from M_1. The purpose of this paper is to show that this construction extends to all the L_p-spaces, $1 < p < \infty$. The L_∞ case was investigated by Ubhaya [9, 10].

2. BEST MONOTONE APPROXIMATION IN $L_p[0, 1]$ FOR $1 < p < \infty$

Let $f \in L_p[0, 1]$ for $1 < p < \infty$. We wish to find g^* nondecreasing and in $L_p[0, 1]$ such that

$$\int_0^1 |f - g^*|^p \leqslant \int_0^1 |f - g|^p \qquad \text{for all such } g.$$

From duality [5], g^* best approximates f in the above sense if and only if

$$\int_0^1 (g^* - g)(f - g^*)|f - g^*|^{p-2} \geqslant 0$$

for all nondecreasing g in $L_p[0, 1]$.

0021-9045/90 $3.00

We now establish a constructive solution to this problem.

DEFIITION 1. For $f \in L_p[0, 1]$, $1 < p < \infty$, and any real c let

$$\phi_c = (f - c)|f - c|^{p-2}, \tag{1}$$

$$k_c(x) = \int_0^x \phi_c, \qquad 0 \leqslant x \leqslant 1. \tag{2}$$

$$m_c = \min\{k_c(x): 0 \leqslant x \leqslant 1\}, \tag{3}$$

and

$$x(c) = \max\{x: k_c(x) = m_c\}. \tag{4}$$

LEMMA 1. *$x(c)$ is nondecreasing in c.*

Proof. First we establish that $\phi_c(x) > \phi_d(x)$ for $c < d$. Let $e_c = f - c$. Then $e_c(x) > e_d(x)$ for $c < d$.

If $e_c(x) > e_d(x) \geqslant 0$, then

$$\phi_c(x) = e_c^{p-1}(x) > e_d^{p-1}(x) = \phi_d(x).$$

If $e_c(x) \geqslant 0 > e_d(x)$, then $\phi_c(x) \geqslant 0 > \phi_d(x)$.

If $0 > e_c(x) > e_d(x)$, then $|e_c(x)| < |e_d(x)|$ and

$$\begin{aligned} -\phi_c(x) &= -e_c(x)|e_c(x)|^{p-2} \\ &= |e_c(x)|^{p-1} \\ &< |e_d(x)|^{p-1} \\ &= -e_d(x)|e_d(x)|^{p-2} \\ &= -\phi_d(x). \end{aligned}$$

Next assume to the contrary that $x(c) > x(d)$ for some $c < d$. Then,

$$\begin{aligned} k_c(x(c)) &= \int_0^{x(c)} \phi_c \\ &= \int_0^{x(d)} \phi_c + \int_{x(d)}^{x(c)} \phi_c \\ &= k_c(x(d)) + \int_{x(d)}^{x(c)} \phi_c \\ &> k_c(x(d)) + \int_{x(d)}^{x(c)} \phi_d \\ &= k_c(x(d)) + k_d(x(c)) - k_d(x(d)) \\ &> k_c(x(d)), \end{aligned}$$

by the definition of $m_d = k_d(x(d))$. This contradicts the definition of $x(c)$.

In the following lemma, as usual $x(-\infty)$ and $x(+\infty)$ denote respectively $\lim_{t\to-\infty} x(t)$ and $\lim_{t\to-\infty} x(t)$.

LEMMA 2. (a) $x(-\infty)=0$, (b) $x(+\infty)=1$.

Proof. The proofs of (a) and (b) are similar. Thus we present only part (a).

Since $k_c(0)=0$, it suffices to show that for any x satisfying $0<x\leqslant 1$, $\liminf_{c\to-\infty} k_c(x)>0$.

For any $c<0$ define the set $E_c=\{x\in[0,1]:f(x)<c\}$,

and let E_c^c denote the complement of E_c in $[0,1]$. Then,

$$|c|^p\,\mu\{E_c\}\leqslant\int_{E_c}(-f)^p\leqslant\|f\|_p^p,$$

where μ denotes Lebesgue measure. Thus,

$$\mu\{E_c\}\leqslant\|f\|_p^p/|c|^p.$$

Next consider $E_c(x)\equiv E_c\cap[0,x]$:

$$|f-c|^{p-1}\leqslant\gamma_p\{|f|^{p-1}+|c|^{p-1}\},$$

where

$$\gamma_p=\max\{1,2^{p-2}\}.$$

Therefore,

$$\begin{aligned}\left|\int_{E_c(x)}(f-c)|f-c|^{p-2}\right|&\leqslant\int_{E_c(x)}|f-c|^{p-1}\\&\leqslant\gamma_p\left\{\int_{E_c(x)}|f|^{p-1}+|c|^{p-1}\mu\{E_c\}\right\}\\&\leqslant\frac{1}{|c|}\gamma_p\left\{\int_{E_c(x)}|f|^p+|c|^p\mu\{E_c\}\right\}\\&\leqslant\frac{2\gamma_p\|f\|_p^p}{|c|}.\end{aligned}$$

Thus,

$$\lim_{c\to-\infty}\int_{E_c(x)}(f-c)|f-c|^{p-2}=0.$$

Finally, consider $E_c^c(x) = E_c^c \cap [0, x]$. Since $\lim_{c \to -\infty} \mu\{E_c^c(x)\} = x$, we can choose $\underline{c}$ so that $\mu\{E_{\underline{c}}^c(x)\} > x/2$. Then, for $c < \underline{c}$

$$(f-c)|f-c|^{p-2} = ((f-\underline{c}) + (\underline{c}-c))|(f-\underline{c}) + (\underline{c}-c)|^{p-2}$$
$$> (f-\underline{c})|f-\underline{c}|^{p-2} \qquad \text{on } E_{\underline{c}}^c.$$

Also, $E_{\underline{c}}^c \subseteq E_c^c$ for $c < \underline{c}$, and therefore since $\mu\{E_{\underline{c}}^c(x)\} > x/2 > 0$

$$\int_{E_c^c(x)} (f-c)|f-c|^{p-2} \geqslant \int_{E_{\underline{c}}^c(x)} (f-c)|f-c|^{p-2}$$
$$\geqslant \int_{E_{\underline{c}}^c(x)} (f-\underline{c})|f-\underline{c}|^{p-2} > 0.$$

Therefore, for any x satisfying $0 < x \leqslant 1$,

$$\liminf_{c \to -\infty} \int_{E_c^c(x)} (f-c)|f-c|^{p-2} > 0,$$

and thus since

$$\int_0^x (f-c)|f-c|^{p-2} = \int_{E_c^c(x)} (f-c)|f-c|^{p-2}$$
$$+ \int_{E_c(x)} (f-c)|f-c|^{p-2}$$

we can conclude that

$$\liminf_{c \to -\infty} \int_0^x (f-c)|f-c|^{p-2} > 0.$$

The following lemma shows that $x(c)$ is continuous from the right. As usual $x(c+)$ denotes $\lim_{t \to c+} x(t)$.

LEMMA 3. $x(c+) = x(c)$.

Proof. For $\delta > 0$

$$k_{c+\delta}(x(c+\delta)) \leqslant k_{c+\delta}(x(c))$$
$$= \int_0^{x(c)} \phi_{c+\delta}$$
$$\leqslant \int_0^{x(c)} \phi_c$$
$$= k_c(x(c))$$
$$= m_c.$$

Letting $\delta \to 0+$ we obtain

$$k_c(x(c+)) = \int_0^{x(c+)} \phi_c \leqslant m_c.$$

By the definition of m_c, $k_c(x(c+)) \geqslant m_c$. Thus $k_c(x(c+)) = m_c$, and, therefore, $x(c+) \leqslant x(c)$. Since $x(c)$ is nondecreasing, it follows that $x(c+) = x(c)$.

In general, $x(c)$ may be discontinuous. If

$$x(c-) < x(c+) = x(c),$$

where $x(c-)$ denotes $\lim_{t \to c-} x(t)$, then we say c is a jump for $x(\cdot)$.

Locating the jumps for $x(\cdot)$ will enable us to define the following approximation g^* which we shall prove to be the best nondecreasing L_p approximation to $f \in L_p[0, 1]$.

DEFINITION 2. Since $x(\cdot)$ is nondecresing and right continuous, by Lemma 2 each $t \in (0, 1)$ is in some interval $[x(c-), x(c)]$. Thus, we define a function $g^*(t)$ on $(0, 1)$ by

$$\begin{gathered} \text{if } t = x(c) \text{ for some real } c, \text{ let} \\ g^*(t) = \inf\{u : x(u) = x(c)\}, \end{gathered} \tag{5}$$

$$\begin{gathered} \text{if } c \text{ is a jump point for } x(\cdot) \text{ and } x(c-) \leqslant t < x(c), \\ \text{let } g^*(t) = c. \end{gathered} \tag{6}$$

LEMMA 4. *$g^*(t)$ is nondecreasing on* $(0, 1)$.

Proof. Let $\{c_i\}$ be the set of all jump points of $x(c)$, and let $t_1 < t_2$.

If $t_1 = x(c)$ and $t_2 = x(u)$, then $c < u$ since $x(\cdot)$ is nondecreasing. By definition, $g^*(t_1) \leqslant g^*(t_2)$.

If $t_1 = x(c)$ and $x(c_i-) \leqslant t_2 < x(c_i)$ for some i, then $c < c_i$. It follows that $g^*(t_1) \leqslant c < c_i = g^*(t_2)$.

Suppose there exist i, j such that $x(c_{j-}) \leqslant t_1 < x(c_j)$ and $x(c_i-) \leqslant t_2 < x(c_i)$. If $i = j$, then $t_1 = c_j = g^*(t_1) = g^*(t_2)$. If $i \neq j$ and if $c_j > c_i$, then $x(c_i) \leqslant x(c_j-)$, which contradicts $t_1 < t_2$. Hence $c_j \leqslant c_i$, and $g^*(t_1) \leqslant g^*(t_2)$.

Finally, suppose that $x(c_i-) \leqslant t_1 < x(c_i)$ for some i and $t_2 = x(c)$. Then $c_i \leqslant c$, and $g^*(t_1) \leqslant g^*(t_2)$.

LEMMA 5. *Let*

$$A_p = \left\{ x \in (0, 1) : \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} |f(t) - f(x)|^p = 0 \right\}.$$

Then, $\mu\{A_p\} = 1$.

Proof. Let $T_\varepsilon f(x) = (1/2\varepsilon)\int_{x-\varepsilon}^{x+\varepsilon} |f(t) - f(x)|^p\,dt$ and let $Tf(x) = \limsup_{\varepsilon\to 0+} T_\varepsilon f(x)$. Pick $g \in C[0, 1]$ such that $\|f-g\|_p < 1/n$. By the continuity of g, $Tg = 0$.

Let $h = f - g$. Then, $h \in L_p[0, 1]$. Also, since $1 < p < \infty$

$$T_\varepsilon h(x) \leqslant 2^{p-1}\left(\frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} |h(t)|^p\,dt + |h(x)|^p\right).$$

Therefore,

$$Th(x) \leqslant 2^{p-1}\left(\limsup_{\varepsilon\to 0} \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} |h(t)|^p\,dt + |h(x)|^p\right)$$

and thus on $[0, 1]$

$$Th \leqslant 2^{p-1}(Mh^p + |h|),$$

where M is the maximal function defined for all $F \in L_1[0, 1]$ by

$$(MF)(x) = \sup_{0<\varepsilon<\infty} \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} |F(t)|\,dt.$$

Now,

$$T_\varepsilon f \leqslant 2^{p-1}(T_\varepsilon g + T_\varepsilon h).$$

Therefore,

$$Tf \leqslant 2^{p-1}(Tg + Th) = 2^{p-1}Th \leqslant 4^{p-1}(Mh^p + |h|^p).$$

Thus, for any $y > 0$,

if $Mh^p \leqslant 4^{1-p}y$ and $|h|^p \leqslant 4^{1-p}y$, then $Tf \leqslant 2y$.

Therefore, $\{Tf > 2y\} \subseteq \{Mh^p > 4^{1-p}y\} \cup \{|h|^p > 4^{1-p}y\}$, where each of the three sets in this relationship denotes the subset of $[0, 1]$ which satisfies the respective inequality. By Theorem 7.5 and inequality (5), p. 138, of Rudin [7],

$$\mu\{Mh^p > 4^{1-p}y\} \leqslant 3\cdot 4^{p-1}y^{-1}\|h^p\|_1 \leqslant 3\cdot 4^{p-1}y^{-1}\|h\|_p^p$$

and

$$\mu\{|h|^p > 4^{1-p}y\} \leqslant 4^{p-1}y^{-1}\|h\|_p^p.$$

Therefore,

$$\mu\{Tf>2y\}\leqslant 4^p y^{-1}/n^p,$$

and since n is arbitrary,

$$\mu\{Tf>2y\}=0.$$

Furthermore, since $y>0$ is also arbitrary,

$$\mu\{Tf>0\}=0.$$

Note. This proof parallels the cited results in Rudin [7].

LEMMA 6. *If $x(c)\in A_p$ as defined in Lemma 5 then*

(a) $f(x(c))=c$, *and*

(b) $g^*(x(c))=c$.

Proof. (a) Let $x(c)\in A_p$ and assume $f(x(c))>c$. Then by the definition of A_p

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{x(c)-\varepsilon}^{x(c)} |f(y)-f(x(c))|^p\,dy=0.$$

For any $\delta>0$, let

$$B_\delta=\{y\in[0,1]:|f(y)-f(x(c))|<\delta\},$$

and let B_δ^c be the complement of B_δ in $[0,1]$.

Also for any $\varepsilon>0$, let $I_\varepsilon=[x(c)-\varepsilon, x(c)]\cap[0,1]$. Since

$$\begin{aligned}\int_{I_\varepsilon} |f(y)-f(x(c))|^p\,dy &\geqslant \int_{B_\delta^c\cap I_\varepsilon} |f(y)-f(x(c))|^p\,dy\\ &\geqslant \delta\int_{B_\delta^c\cap I_\varepsilon} |f(y)-f(x(c))|^{p-1}\,dy\\ &\geqslant \delta^p\mu\{B_\delta^c\cap I_\varepsilon\},\end{aligned}$$

it follows that

$$\lim_{\epsilon\to 0}\frac{1}{\varepsilon}\int_{B_\delta^c\cap I_\varepsilon} |f(y)-f(x(c))|^{p-1}\,dy=0,$$

and

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mu\{B_\delta^c\cap I_\varepsilon\}=0.$$

Thus, letting $\gamma_p = \max\{1, 2^{p-1}\}$,

$$\left|\int_{B_\delta^c \cap I_\varepsilon} (f-c)|f-c|^{p-2}\right|$$

$$\leqslant \int_{B_\delta^c \cap I_\varepsilon} |f-c|^{p-1}$$

$$\leqslant \gamma_{p-1}\int_{B_\delta^c \cap I_\varepsilon} |f-f(x(c))|^{p-1} + \gamma_{p-1}\int_{B_\delta^c \cap I_\varepsilon} |f(x(c))-c|^{p-1}$$

$$= \gamma_{p-1}\int_{B_\delta^c \cap I_\varepsilon} |f-f(x(c))|^{p-1} + \gamma_{p-1}|f(x(c))-c|^{p-1}\mu\{B_\delta^c \cap I_\varepsilon\}$$

and therefore

$$\lim_{\varepsilon\to 0}\left|\frac{1}{\varepsilon}\int_{B_\delta^c \cap I_\varepsilon} (f-c)|f-c|^{p-2}\right|$$

$$\leqslant \gamma_{p-1}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{B_\delta^c \cap I_\varepsilon} |f-f(x(c))|^{p-1}$$

$$+\gamma_{p-1}|f(x(c))-c|^{p-1}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mu\{B_\delta^c \cap I_\varepsilon\}$$

$$=0.$$

Thus,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{B_\delta^c \cap I_\varepsilon} (f-c)|f-c|^{p-2} = 0. \tag{7}$$

Now fix $\delta > 0$ so that $f(x(c)) > c+\delta$. Then, for $y \in B_\delta$,

$$0 < f(x(c)) - \delta - c < f(y) - c < f(x(c)) + \delta - c.$$

Hence,

$$\int_{B_\delta \cap I_\varepsilon} (f-c)|f-c|^{p-2}$$

$$> \begin{cases} \int_{B_\delta \cap I_\varepsilon} (f(x(c))-\delta-c)|f(x(c))+\delta-c|^{p-2}, & 1<p<2 \\ \int_{B_\delta \cap I_\varepsilon} (f(x(c))-\delta-c)|f(x(c))-\delta-c|^{p-2}, & 2\leqslant p \end{cases}$$

$$\equiv Q\mu\{B_\delta \cap I_\varepsilon\}, \qquad \text{where} \quad Q>0.$$

Using (7), it follows that

$$\liminf_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{I_\varepsilon} (f-c)|f-c|^{p-2}$$
$$\geqslant Q \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mu\{B_\delta \cap I_\varepsilon\} = Q > 0.$$

Hence, for $\varepsilon > 0$ and sufficiently small,

$$\int_{x(c)-\varepsilon}^{x(c)} (f-c)|f-c|^{p-2} > 0.$$

Thus $k_c(x(c)-\varepsilon) < k_c(x(c))$, contradicting the definition of $x(c)$.

In a similar way, we get a contradiction if we assume that $f(x(c)) < c$. Hence $f(x(c)) = c$.

(b) If $x(u) = x(c) \in A_p$, then (a) implies that $c = f(x(c)) = f(x(u)) = u$. Thus $\{u: x(u) = x(c)\} = \{c\}$. Therefore $g^*(x(c)) = c$.

LEMMA 7. *If* $x(c-) \leqslant t \leqslant x(c)$, *then*

(a) $\int_{x(c-)}^{t} \phi_c \geqslant 0$, *and*

(b) $\int_{x(c-)}^{x(c)} \phi_c = 0$.

Proof. If $x(c-) = x(c)$, then the lemma holds trivially. Thus we need only consider the case $x(c-) < x(c)$.

Assume that $\int_{x(c-)}^{t} \phi_c < 0$ for some t satisfying $x(c-) < t \leqslant x(c)$. Then for $\delta > 0$ and sufficiently small, $\int_{x(c-\delta)}^{t} \phi_{c-\delta} < 0$. Thus,

$$\begin{aligned} k_{c-\delta}(t) &= \int_0^t \phi_{c-\delta} \\ &< \int_0^{x(c-\delta)} \phi_{c-\delta} \\ &= k_{c-\delta}(x(c-\delta)) \\ &= m_{c-\delta}, \end{aligned}$$

which is a contradiction. Thus (a) is verified.

From (a), $\int_{x(c-)}^{x(c)} \phi_c \geqslant 0$. If $\int_{x(c-)}^{x(c)} \phi_c > 0$, then

$$\begin{aligned} k_c(x(c-)) &= \int_0^{x(c-)} \phi_c \\ &< \int_0^{x(c)} \phi_c \\ &= k_c(x(c)) \\ &= m_c, \end{aligned}$$

which contradicts the definition of $x(c)$. Thus (b) is verified.

LEMMA 8. $g^* \in L_p[0, 1]$.

Proof. Let $\{c_i\}$ be the discontinuities of $x(c)$. For $t \in [x(c_i-), x(c_i)]$, $g^*(t) = c_i$. By Lemma 5,

$$\int_{x(c_i-)}^{t} \phi_{c_i} \geqslant 0 \qquad \text{and} \qquad \int_{x(c_i-)}^{x(c_i)} \phi_{c_i} = 0.$$

Thus, by duality, $g^* \equiv c_i$ is the best constant approximation to f on $[x(c_i-), x(c_i)]$.

Let A_p be the set defined in Lemma 5. For $t \in A_p$ either $t = x(c)$ for some c, in which case $f(x(c)) = c = g^*(x(c))$, or $x(c_i-) \leqslant t < x(c_i)$ for some i.

If $i \neq j$, then $(x(c_i-), x(c_i)) \cap (x(c_j-), x(c_j)) = \varnothing$. Hence,

$$\begin{aligned} \int_0^1 |f - g^*|^p &= \int_{\bigcup_i (x(c_i-), x(c_i))} |f - c_i|^p \\ &\leqslant \int_{\bigcup_i (x(c_i-), x(c_i))} |f|^p \leqslant \|f\|_p^p. \end{aligned}$$

Thus $f - g^* \in L_p[0, 1]$, and, therefore, $g^* \in L_p[0, 1]$.

We can now show that g^* is the best nondecreasing L_p approximation to f from $L_p[0, 1]$.

THEOREM. *If $f \in L_p[0, 1]$, then g^*, as given in Definition 2, is the unique best nondecreasing L_p approximation to f from $L_p[0, 1]$.*

Proof. Let A_p be as in Lemma 5, and let $\{c_i\}$ be the discontinuities of $x(c)$. By Lemma 5, A_p has measure one. Let $A_p^1 = A_p \setminus \bigcup_i (x(c_i-), x(c_i))$. Define $\phi_{g^*} = (f - g^*)|f - g^*|^{p-2}$. By Lemma 6, $\phi_{g^*} = 0$ on A_p^1.

Now define $r(t)=\int_0^t \phi_{g^*}$. If $t=x(c)$, then

$$\begin{aligned} r(t) &= \int_{A_p \cap [0,t]} \phi_{g^*} \\ &= \sum_{c_i \leqslant c} \int_{x(c_i-)}^{x(c_i)} \phi_{g^*} \\ &= \sum_{c_i \leqslant c} \int_{x(c_i-)}^{x(c_i)} \phi_{c_i} \\ &= 0, \qquad \text{by Lemma 7.} \end{aligned}$$

If $x(c_{j-}) \leqslant t < x(c_j)$, then

$$\begin{aligned} r(t) &= \int_{x(c_j-)}^{t} \phi_{g^*} \\ &= \int_{x(c_j-)}^{t} \phi_{c_j} \\ &\geqslant 0, \qquad \text{by Lemma 7.} \end{aligned}$$

We also have

$$r(1)=\sum_i \int_{x(c_i-)}^{x(c_i)} \phi_{c_i}=0.$$

Thus $r(t) \geqslant 0$.

Next we note that

$$\begin{aligned} \int_0^1 g^*\phi_{g^*} &= \int_{A_p} g^*\phi_{g^*} \\ &= \sum_i \int_{x(c_i-)}^{x(c_i)} c_i\phi_{c_i} = 0. \end{aligned}$$

Now let g be a nondecreasing function in $L_p[0,1]$. Define

$$g_n(x)=\begin{cases} g(x), & -n \leqslant g(x) \leqslant n \\ -n, & g(x) < -n \\ n, & n < g(x). \end{cases}$$

Then, pointwise, $g_n \to g$, $g_n \phi_{g^*} \to g\phi_{g^*}$, and $|g_n \phi_{g^*}| \leqslant |g\phi_{g^*}|$. By the Lebesgue Dominated Convergence Theorem,

$$\int_0^1 g_n \phi_{g^*} \to \int_0^1 g\phi_{g^*}$$

and, using integration by parts,

$$\int_0^1 g_n \phi_{g^*} = -\int_0^1 r(t)\, dg_n \leqslant 0,$$

since $r(t) \geqslant 0$ and g_n is nondecreasing. Therefore

$$\int_0^1 g\phi_{g^*} \leqslant 0 = \int_0^1 g^* \phi_{g^*}.$$

Thus, g^* is the best L_p nondecreasing approximation to f.

Remarks. (a) If $f \in C[0, 1]$, then Lemma 6 implies that $x(c)$ is strictly increasing, and f is nondecreasing on

$$\{x(c) : 0 < x(c) < 1\}.$$

Furthermore the definition of g^* simplifies to

$$g^*(t) = \begin{cases} c_i, & x(c_i -) \leqslant t \leqslant x(c_i) \\ f(t), & \text{elsewhere}, \end{cases}$$

where, as before, $\{c_i\}$ denotes the set of jumps of $x(c)$.

(b) The method used in the proof of the theorem can be used in the proof of Lemma 8 to show that $g^* \equiv c_i$ is the best nondecreasing approximation to f on $[x(c_i -), x(c_i)]$.

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Printed in Belgium